Robust macroscopic quantum correlations without complex encodings

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One of the main challenges for the manipulation and storage of macroscopic entanglement is its fragility under noise. We present a simple recipe for the systematic enhancement of the resistance of multipartite entanglement against any local noise with a privileged direction in the Bloch sphere. For the case of exact local dephasing along any given basis, and for all noise strengths, our prescription grants full robustness: even states with exponentially decaying entanglement are mapped to states whose entanglement is constant. In contrast to previous techniques resorting to complex logical-qubit encodings, such enhancement is attained simply by performing local unitary rotations before the noise acts. The scheme is therefore highly experimentally-friendly, as it brings no overhead of extra physical qubits to encode logical ones. In addition, we show that, apart from entanglement, the resiliences of the relative entropy of quantumness and the usefulness as resources for practical tasks such as metrology and nonlocality-based protocols are equivalently enhanced.

Introduction.— Multipartite quantum correlations in composite systems subject to local noise are in general extremely fragile, typically decaying the faster the larger the number N of particles. For instance, one of the most important genuinely multipartite entangled states is the Greenberger-Horne-Zeilinger (GHZ) state [1]. It represents coherent superpositions of the kind of the celebrated Schrödinger's cat state [2]. Unfortunately, under the action of local noise, its entanglement decays exponentially fast, with a decay-rate that grows proportionally to N [3–5]. Thus, the system is very rapidly taken into, or close to, a separable mixture. This exponential fragility entails serious drawbacks for the practical applicability of GHZ states as resources for quantum information processing in realistic scenarios. For example, for any noise strength, the quantum gain provided by GHZ entanglement in parameter estimation [6] or distributed-computing scenarios [7–9] vanishes almost instantaneously already for moderate system size.

A possible way to enhance the robustness of quantum correlations is to encode logical qubits into error-correction codewords, consisting of entangled states of many physical qubits [4, 10]. For instance, for small noise strengths p, the decay rate of logical GHZ entanglement under local white noise can be decreased exponentially with the number of physical qubits in the codeword when the codewords are themselves GHZ states [10]. This is remarkable because the enhancement is achieved passively, *i.e.* without any active error correction. However, there is a price to pay in experimental overhead: For the logical state to achieve full entanglement robustness – in the sense that its logical entanglement becomes independent on N–, each logical qubit requires a number of genuine-multipartite entangled physical qubits that scales logarithmically with the number of logical qubits.

With the maximally mixed state as its only steady state, local white noise (local depolarization) is the most detrimental type of local noise. Nevertheless, in many realistic situations the noise can be assumed, up to good approximation, to posses privileged directions in the Bloch sphere, including pure states as steady states. This is

the case for instance in many experiments with atomic or ionic qubits, where the dominant source of noise is dephasing from magnetic-field and laser-intensity fluctuations, and from spontaneous emissions during Raman couplings [11]. Another example is provided by birefringent polarization-mantaining optical fibers [12], where mechanical stress and temperature induce index-refraction fluctuations that dephase polarization qubits. In the former case, the privileged noise direction is that of the quantization axis defined by the magnetic field; while in the latter, that of the linear polarizations along the ordinary and extraordinary axes of the fiber.

In this work we study the action on graph states of local noisy channels with an approximately well-defined privileged basis. Graph states constitute a family of genuine multi-qubit entangled states with remarkable applications [13]. Relevant examples thereof are the previously mentioned GHZ state, or the cluster state, which allows for measurement based quantum computation [14]. We introduce an experimentally friendly recipe, consisting of local unitary rotations before the noise acts, to enhance the resistance of graph-state quantum correlations. Remarkably, and despite its simplicity, for exact dephasing this prescription supplies the states with full robustness: It gives an N-independent lower bound for the decay of graph-state entanglement. In particular, the exponentially fragile entanglement of GHZ states is enhanced to decay only linearly with p, for all N. In addition, the bound holds for quantum correlations other than entanglement [15] and is robust against mixedness in the initial states. Finally, for GHZ states, we show that the local-unitary protection resists noise deviations from exact dephasing, and that the enhancement applies also to the usefulness for physical tasks such as metrology [6] and distributed-computing [7, 8] protocols.

Enhancement of robustness of graph-state quantum correlations.— We consider local completely-positive

trace-preserving channels $\mathcal{E}$, defined on any state ϱ as

$$\mathcal{E}(\varrho) \doteq (1 - \frac{p}{2})\varrho + \frac{p}{2}(\alpha_X X \varrho X$$
$$+ \alpha_Y Y \varrho Y + \alpha_Z Z \varrho Z), \qquad (1)$$

where X, Y and Z are respectively the first, second and third Pauli matrices in the computational basis $\{|0\rangle, |1\rangle\}$. Parameters $0 \leq \alpha_X, \alpha_Y, \alpha_Z \leq 1$ satisfy the normalization condition $\alpha_X + \alpha_Y + \alpha_Z = 1$. The composite N-qubit map Λ is given by the single-qubit map composition $\Lambda(\varrho) \doteq \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes ... \mathcal{E}_N(\varrho)$, where $\mathcal{E}_k$, with $1 \leq k \leq N$, corresponds to map (1) acting on the k-th qubit. The focus of our attention throughout is in the situations where $\alpha_Z >> \alpha_X, \alpha_Y$, so that Λ is close to local phase-damping map Λ^{PD}, which corresponds to $\alpha_Z = 1$. Probability $0 \leq p \leq 1$ measures the noise strength and gives also a convenient parametrization of time: $p = 0$ refers to the initial time $t = 0$ and $p = 1$ refers to the asymptotic $t \to \infty$ limit. Note that the $1/2$ factors in (1) are such that an exact fully-dephasing channel appears at $p = 1$ and $\alpha_z = 1$.

Let us begin by the phase-damping channel $\alpha_Z = 1$. We focus first on GHZ states

$$|\Phi_+{}^N\rangle \doteq \frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes N} + |1\rangle^{\otimes N}\right). \qquad (2)$$

Under Λ^{PD}, all the entanglement in (2) decays (at slowest) exponentially with N, as $(1 - p)^N$ [5]. We show next that, for a fixed p, the entanglement of

$$|\Phi_{+T}{}^N\rangle \doteq H^{\otimes N}|\Phi_+{}^N\rangle = |+\rangle^{\otimes N} + |-\rangle^{\otimes N}, \qquad (3)$$

under Λ^{PD} is independent on N. Operator H stands for the Hadamard-gate rotation, defined by $H|0\rangle \doteq |+\rangle$ and $H|1\rangle \doteq |-\rangle$, with $|\pm\rangle \doteq \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. Transversal states (3) are thus local-unitarily equivalent to (2), possessing therefore the same amount and type of entanglement.

We consider here an arbitrary entanglement monotone E, i. e. any function of ϱ which is non-increasing under local operations and classical communication. In App. A however, we extend the treatment to the relative entropy of quantumness [15], which is not an entanglement monotone. Notice first that a single-qubit Z measurement on $|\Phi_{+T}{}^N\rangle$ leaves the remaining $N - 1$ qubits in state $|\Phi_{+T}{}^{N-1}\rangle$, for outcome 0, or in $|\Phi_{-T}{}^{N-1}\rangle \doteq |+\rangle^{\otimes N} - |-\rangle^{\otimes N}$, for outcome 1. State $|\Phi_{-T}{}^{N-1}\rangle$ is local-unitarily equivalent to $|\Phi_{+T}{}^{N-1}\rangle$. Now, since it commutes with Λ^{PD}, a Z measurement on $\Lambda^{PD}(|\Phi_{+T}{}^N\rangle)$ leaves the remaining qubits in $\varrho_{+T}^{N-1} \doteq \Lambda^{PD}(|\Phi_{+T}{}^{N-1}\rangle)$ or $\varrho_{-T}^{N-1} \doteq \Lambda^{PD}(|\Phi_{-T}{}^{N-1}\rangle)$. Furthermore, it is immediate to see that ϱ_{+T}^{N-1} and ϱ_{-T}^{N-1} are also local-unitarily equivalent. From this, and the monotonicity of E under local measurements, it follows that $E(\varrho_{+T}^N) \geq E(\varrho_{+T}^{N-1})$. Iterating this reasoning $N - 2$ times, one obtains that

$$E\left(\Lambda^{PD}(|\Phi_{+T}{}^N\rangle)\right) \geq E\left(\Lambda^{PD}(|\Phi_{+T}{}^{N-1}\rangle)\right) \geq$$
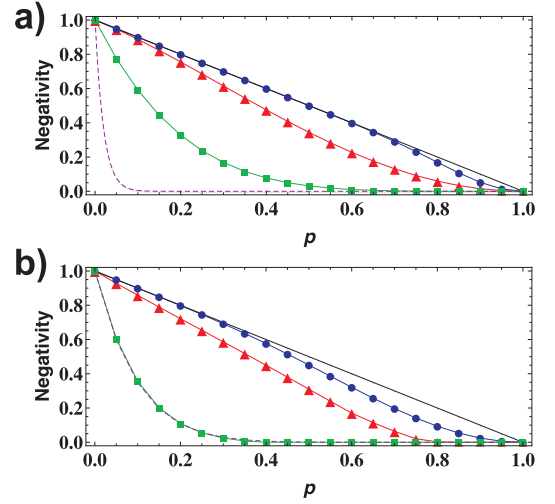$$... \geq E\left(\Lambda^{PD}(|\Phi_{+T}{}^2\rangle)\right). \qquad (4)$$

FIG. 1. (Color online) **a.** Negativities of bipartitions "one qubit versus the rest", as a function of p, for different initial pure states under local dephasing. Already for $N = 5$, transversal GHZ state $|\Phi_{+T}{}^5\rangle$ (red triangles) displays a considerably higher robustness than bare GHZ state $|\Phi_+{}^5\rangle$ (green squares). As N goes to infinity, the negativity corresponding to $|\Phi_{+T}{}^N\rangle$ tends to $1 - p$ (thin black line). However, for any finite N there is always a p up to which this negativity is approximately equal to the N-independent $1 - p$ curve; and the smaller the N, the smaller the p for which this happens. For $N = 50$ (blue circles) for instance, this happens for $p \approx 0.65$. The purple dashed line corresponds to $|\Phi_+{}^{50}\rangle$. **b.** Same negativities, for $N = 10$, but with a deviation $\epsilon = 0.05$ from exact dephasing. Transversal GHZ states (red triangles) are still exponentially more robust than bare GHZ states (green squares). The negativity of transversal GHZ states for the case $\epsilon = 0$ (blue circles) is always on top, while the one of bare GHZ states for $\epsilon = 0$ (purple dashed) is almost identical to that of bare GHZ states for $\epsilon = 0.05$. For comparison, the linear decay $1 - p$ is also shown (thin solid black)

That is, transversal states (3) possess at least as much resistance as two-qubit state $|\Phi_{+T}{}^2\rangle$, for all $N \geq 2$. In App. B, we show that robustness law (4) holds not only for GHZ states but actually for arbitrary graph states, even initially in the presence of global white noise: Their entanglement decay is bounded from below by the decay of the corresponding two-qubit graph state. The validity of the bound again follows from the fact that any connected N-qubit graph state can be mapped into a connected graph state of $N - 1$ qubits by a single-qubit measurement.

As a quantitative assessment, we probe bound (4) with a simple-to-calculate entanglement monotone: the negativity [16]. In addition, to test their robustness against deviations from exact dephasing, we allow for maps Λ with arbitrary α_x, α_y, and α_z. Specifically, we calculate analytically the negativity $\mathcal{N}$ of any bipartition "one qubit versus the rest" of $\Lambda(|\Phi_{+T}{}^N\rangle)$. The smaller $\mathcal{N}$ is, the closer the state is to featuring no genuine N-qubit entanglement [3–5]. Due to the GHZ-diagonal structure

(see for instance [5]) of $\Lambda(|\Phi_+{}^N_T\rangle)$, the calculation is enormously simplified, and reduces essentially to diagonalizing 2^{N-1} matrices of dimensions 2×2. One obtains

$$\mathcal{N} = \sum_{\mu=0}^{\lfloor \frac{N-1}{2} \rfloor} \binom{N-1}{\mu} \Big(\max[0, f^-_{\mu+1} - f^+_\mu]$$
$$+ \max[0, f^-_\mu - f^+_{\mu+1}] \Big), \qquad (5)$$

where $f^\pm_\mu \doteq (\frac{p}{2}\alpha_Z \pm \frac{p}{2}\alpha_Y)^\mu \big((1-\frac{p}{2}) \pm \frac{p}{2}\alpha_X\big)^{N-\mu} + (\frac{p}{2}\alpha_Z \pm \frac{p}{2}\alpha_Y)^{N-\mu}\big((1-\frac{p}{2}) \pm \frac{p}{2}\alpha_X\big)^\mu$ and $\lfloor \frac{N-1}{2} \rfloor \doteq N/2 - 1$, for N even, or $\lfloor \frac{N-1}{2} \rfloor \doteq (N-1)/2$, for N odd.

For the particular case $\alpha_Z = 1$ addressed by (4), the negativity (5) simplifies (see App. C) to

$$\mathcal{N}_{(\alpha_Z=1)} = (1-p) \sum_{\mu=0}^{\lfloor \frac{N'}{2} \rfloor} \binom{N'}{\mu} \Big[\frac{p^\mu}{2}(1-\frac{p}{2})^{N'-\mu}$$
$$- \frac{p}{2}^{N'-\mu}(1-\frac{p}{2})^\mu \Big], \qquad (6)$$

with $N' \doteq N - 1$. In App. C we show that $\mathcal{N}_{(\alpha_Z=1)} \to 1 - p$ as $N \to \infty$, for all $0 \le p < 1$. In addition, for all $0 \le p \le 1$, we observe that the bound on $\mathcal{N}_{(\alpha_Z=1)}$ given by (4) is too conservative, as it is not tight. Furthermore, the limit $1-p$ is approached the faster the smaller p. This can be appreciated in Fig. 1 a), where $\mathcal{N}_{(\alpha_Z=1)}$ for states (2) and (3) under local-dephasing map Λ^{PD} is plotted as function of p and for different N.

In turn, for the generic case $\alpha_Z = 1 - \epsilon \le 1$, with $\epsilon \doteq \alpha_X + \alpha_Y$ the deviation from exact dephasing, we focus on the weak-noise regime. For sufficiently small p, one can ignore all terms not linear in p and approximate negativity (5) as $\mathcal{N} \approx 1 - p(N\epsilon + 1 - \epsilon) \approx (1-p)^{N\epsilon+1-\epsilon}$. While this is no longer an N-independent behavior, it clearly constitutes a remarkable robustness enhancement in comparison to $(1-p)^N$, specially in the limit $\epsilon << 1$ [see Fig. 1 b)]. In addition, numerical tests show that the approximation actually holds up to relatively large p. For example, with $\epsilon = 0.1$ and $N = 20$, the approximation by the exponential above is excellent up to $p \approx 0.2$. Furthermore, this in turn tends to $\mathcal{N} \approx (1-p)^{N\epsilon}$ as N increases, an exponential decay as that of (2) but with the exponent damped by the factor ϵ. In all cases, the less the noise deviates from exact dephasing ($\alpha_Z = 1$), the slower the decay of entanglement is.

We emphasize that the enhancement is achieved without any overhead in complex logical-qubit encodings, just through local Hadamard rotations. Interestingly, these rotations correspond to a qubit-basis transversal to the one privileged by to the noise. In particular, for GHZ entanglement, the robust qubit-basis is exactly the one orthogonal to that defined by the only pure states immune to the noise. Numerical optimizations up to $N = 5$ show that no other single-qubit basis yields slower decay of negativity than that of (3), although other single-qubit bases attain the same decay (up to numerical precision).
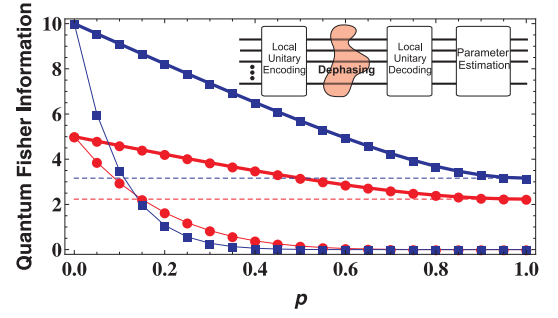

FIG. 2. (Color online) If, before local dephasing acts (inset), GHZ states are encoded into transversal qubit-bases, and then decoded before the parameter estimation, their usefulness for the estimation is deteriorated exponentially less. The resulting (square root of the) Fisher information $\sqrt{\mathcal{F}^{PD}_T}$ (thick lines) is exponentially above that of bare GHZ states, $\sqrt{\mathcal{F}^{PD}}$ (thin lines). Blue squares correspond to $N = 10$ qubits, whereas red circles to $N = 5$. Dashed thin lines delimit the maxima of $\sqrt{\mathcal{F}}$ over the separable states. Notice that while $\sqrt{\mathcal{F}^{PD}_T}$ lies above the classical limit throughout the dynamics, $\sqrt{\mathcal{F}^{PD}}$ crosses it exponentially fast.

Quantum metrology with dephased resources.— For mixed states, higher entanglement does not necessarily imply better performance at fulfilling some physical task. In what follows, we show that our local-unitary protection enhances also the robustness against local dephasing of GHZ states as resources for phase estimation [6].

We consider a Hamiltonian $\hat{H} = \lambda \sum_{k=1}^N Z_k$, with unknown parameter λ. The associated unitary evolution operator over a time t is $U_\phi = e^{-it\hat{H}}$, with $\phi \doteq \lambda t$ the phase to be estimated. A generic N-qubit probe state ϱ accordingly transforms as $\varrho \to \varrho_\phi \doteq U_\phi \varrho U_\phi^\dagger$. The statistical deviation $\delta\phi$ in the estimation can be bounded as $\delta\phi \ge 1/\left(2\sqrt{\nu \mathcal{F}(\varrho_\phi)}\right)$ [17]. Here, ν represents the number of runs of the estimation, and $\mathcal{F}(\varrho_\phi)$ is the quantum Fisher Information [18] of ϱ_ϕ, which measures how much information about ϕ can be extracted from ϱ_ϕ. If ϱ_ϕ is separable, $\mathcal{F}(\varrho_\phi)$ is always limited as $\mathcal{F}(\varrho_\phi) \le N$. However, entangled states can reach the maximal value $\mathcal{F}_{max} = N^2$, which yields a quadratic gain in precision. This is the maximal gain compatible with the uncertainty principle and is therefore known as the Heisenberg limit.

The Fisher information of locally dephased GHZ state $\varrho = \Lambda^{PD}(|\Phi_+{}^N\rangle)$ is $\mathcal{F}^{PD} = N^2(1-p)^{2N}$. Clearly, the quantum gain is obliterated by decoherence exponentially fast. However, if before dephasing takes place one locally rotates the resource GHZ state to (3), and then, before the estimation, undoes the rotation, the resulting Fisher information $\mathcal{F}^{PD}_T \doteq \mathcal{F}\big(H^{\dagger \otimes N} \Lambda^{PD}(|\Phi_+{}^N_T\rangle) H^{\otimes N}\big)$ is

$$\mathcal{F}^{PD}_T = N^2(1-p)^2 + 4N\big(1-\frac{p}{2}\big)\frac{p}{2}. \qquad (7)$$

The robustness-enhancement is thus exponential also for the accuracy in phase estimations (see Fig. 2).
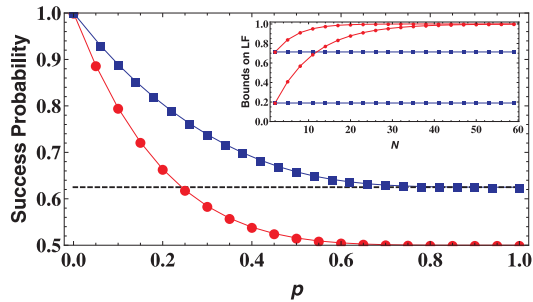
Nonlocal computations with dephased resources.— Fi-

FIG. 3. (Color online) Probabilities of success for the Mermin-Klyshko CCP as a function of p and for $N = 5$ qubits. Blue squares correspond to the correlations in $\Lambda^{PD}(|\Phi_{+_T}^N\rangle)$, whereas red circles to those in $\Lambda^{PD}(|\Phi_+^N\rangle)$. The thin dashed line in turns delimits the maximum classical probability of success. With the local-unitary protection there is a quantum gain throughout the noisy dynamics, while the dephased bare-GHZ correlations enter the classical domain exponentially fast (not shown). This behavior is also reflected by the local fraction LF of the correlations. The inset shows lower and upper bounds of LF, as a function of N and for $p = 0.1$, both for $\Lambda^{PD}(|\Phi_{+_T}^N\rangle)$ (blue squares) and for $\Lambda^{PD}(|\Phi_+^N\rangle)$ (red circles). While the bounds for the latter tend to unit exponentially fast, those for the former are size-independent.

nally, we show that our local-unitary protection scheme enhances the robustness against local dephasing of the nonlocality of quantum correlations. Specifically, we focus on the performance of dephased GHZ states to assist in solving distributed-computing tasks, also known as communication-complexity problems (CCPs) [7, 8].

In the considered CCPs, N distant users, assisted by some correlations and a restricted amount of communication, must locally calculate the value of a given function f. For every Bell inequality there exists a CCP that can be solved with higher probability of success with nonlocal correlations than with any classical resource if, and only if, the correlations violate the inequality [8]. The probability of success in the CCP is $P_S = (1/2)(1 + \beta_Q/\beta_{NL})$, where β_Q is the Bell violation by the nonlocal correlations in the resource quantum state and β_{NL} the maximum violation over arbitrary nonlocal correlations.

As an example, we consider the CCP associated to the Mermin-Klyshko (MK) inequality for N-bit correlations [19]. We obtain that $\Lambda^{PD}(|\Phi_{+_T}^N\rangle)$ violates the inequality for all $0 \leq p < 1$, and its violation is bigger than that by $\Lambda^{PD}(|\Phi_+^N\rangle)$. This leads to the enhancements of P_S as the one plotted in Fig. 3. Also, in the inset of Fig. 3, we have plotted the dependence with N of simple lower and upper bounds of the local fraction of the correlations, which quantifies the fraction of events describable by a local model [20]. The bounds were calculated as explained in [9]. From these, one can see that while, for large N, the local fraction of dephased bare GHZ states tends to unit exponentially fast, that of transversal GHZ states stays always below a constant value (≈ 0.7).

Conclusions. — There are many relevant situations in which multi-particle entanglement is subject to local noise, *e.g.* the distribution of entangled particles to many distant parties or the storage of these particles into different quantum memories. Here we have focused on the physically relevant case in which the noise has a privileged direction and have provided a simple and experimentally friendly recipe to enhance the robustness of quantum correlations. We have shown that a simple local change of bases, while preserving the correlation properties of the state, significantly improves its robustness. For general graph states, we have derived bounds on the decay entanglement and relative entropy of quantumness that are independent of N. In the case of GHZ states, we have shown not only that the local-unitary encoding neutralizes their exponential decay with the system size, but also that an exponential improvement is still observed when there are deviations from the ideal case. In addition, the robustness of the usefulness of GHZ states as resources for parameter estimation and nonlocal computations is equivalently enhanced.

The enhancement introduces no cost at all in extra particles. We believe that the fact that an exponential enhancement is achieved through such an extremely simple scheme, makes the present passive-protection approach highly relevant to many current experimental platforms.

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Appendix A: Robustness law for the relative entropy of quantumness

The relative entropy of quantumness is a discord-like measure of quantum correlations (see [15] and references therein). That is, it encapsulates entanglement but does not restrict to it. Like all variants of discord, it is not non-increasing under local operations assisted by classical communication (LOCC). Indeed, it is not even non-increasing under general local operations [22]. Still, in this appendix we show that a robustness law equivalent to (4) applies to it too.

The relative entropy of quantumness $Q_S(\varrho)$ of an N-qubit state ϱ is defined [15] as $Q_S(\varrho) \doteq S(\varrho||\xi_{\min})$. The von Neumann relative entropy $S(\varrho||\xi_{\min}) \doteq -\text{Tr}[\varrho \log(\xi_{\min})] + \text{Tr}[\varrho \log(\varrho)]$ between states ϱ and $\xi_{\min}$ measures how distinguishable they are. State $\xi_{\min}$ is the closest classical state to ϱ, in the sense of minimizing the relative entropy with ϱ over all exclusively-classically correlated N-qubit states $\xi \doteq \sum_{i_1...i_N} p_{i_1...i_N} |i_1 \ldots i_N\rangle\langle i_1 \ldots i_N|$, with $p_{i_1...i_N}$ any probability distribution and $\{|i_1 \ldots i_N\rangle\}$ any N-qubit basis. We show in what follows that

$$Q_S\big(\Lambda^{PD}(|\Phi_{+T}^N\rangle)\big) \geq \ldots \geq Q_S\big(\Lambda^{PD}(|\Phi_{+T}^2\rangle)\big), \text{ (A1)}$$

for all $N \geq 2$.

As in the derivation of (4), we consider a single-qubit Z measurement acting on $\Lambda^{PD}(|\Phi_{+T}^N\rangle)$, which leaves the system in state $\frac{1}{2}\big(|0\rangle\langle 0| \otimes \varrho_{+T}^{N-1} + |1\rangle\langle 1| \otimes \varrho_{-T}^{N-1}\big)$. As said, Q_S cannot be guaranteed not to increase under generic local maps. However, for the particular case when the local maps are unital (those mapping the identity operator $\mathbb{1}$ into itself), it was shown in Ref. [22] that Q_S is non-increasing. Thus, since a single-qubit Z measurement is a local unital map, we have that

$$Q_S\big(\Lambda^{PD}(|\Phi_{+T}^N\rangle)\big) \geq$$
$$Q_S\Big(\frac{1}{2}\big(|0\rangle\langle 0| \otimes \varrho_{+T}^{N-1} + |1\rangle\langle 1| \otimes \varrho_{-T}^{N-1}\big)\Big). \text{ (A2)}$$

Besides, by definition it is $Q_S\big(\frac{1}{2}(|0\rangle\langle 0| \otimes \varrho_{+T}^{N-1} + |1\rangle\langle 1| \otimes \varrho_{-T}^{N-1})\big) \doteq S\big(\frac{1}{2}(|0\rangle\langle 0| \otimes \varrho_{+T}^{N-1} + |1\rangle\langle 1| \otimes \varrho_{-T}^{N-1})||\tilde{\xi}_{\min}^N\big)$, with $\tilde{\xi}_{\min}^N$ the closest classical state to $\frac{1}{2}(|0\rangle\langle 0| \otimes \varrho_{+T}^{N-1} + |1\rangle\langle 1| \otimes \varrho_{-T}^{N-1})$. Next, using the definition of the relative entropy in terms of traces, taking the partial trace over the measured qubit, and after a straightforward calculation, we obtain

$$S\Big(\frac{1}{2}\big(|0\rangle\langle 0| \otimes \varrho_{+T}^{N-1} + |1\rangle\langle 1| \otimes \varrho_{-T}^{N-1}\big)||\tilde{\xi}_{\min}^N\Big) =$$
$$\frac{1}{2}\Big[S\Big(\varrho_{+T}^{N-1}||\frac{\langle 0|\tilde{\xi}_{\min}^N|0\rangle}{\text{Tr}[\langle 0|\tilde{\xi}_{\min}^N|0\rangle]}\Big)$$
$$+ S\Big(\varrho_{-T}^{N-1}||\frac{\langle 1|\tilde{\xi}_{\min}^N|1\rangle}{\text{Tr}[\langle 1|\tilde{\xi}_{\min}^N|1\rangle]}\Big)$$
$$- \log\big(2x\big) - \log\big(2(1-x)\big)\Big], \text{ (A3)}$$

with $x \doteq \text{Tr}[\langle 0|\tilde{\xi}_{\min}^N|0\rangle]$ and $1-x \doteq \text{Tr}[\langle 1|\tilde{\xi}_{\min}^N|1\rangle]$.

Now, from (A3) we immediately obtain an explicit form for $\tilde{\xi}_{\min}^N$: It must be

$$\tilde{\xi}_{\min}^N = \frac{1}{2}\big(|0\rangle\langle 0| \otimes \xi_{\min+}^{N-1} + |1\rangle\langle 1| \otimes \xi_{\min-}^{N-1}\big), \text{(A4)}$$

with $\xi_{\min+}^{N-1}$ and $\xi_{\min-}^{N-1}$ the closest classical $(N-1)$-qubit states to ϱ_{+T}^{N-1} and ϱ_{-T}^{N-1}, respectively. This is due to the following observations: (i) Clearly, with this form, both the first and second lines after the equality in (A3) are minimized. (ii) The minimum of $-\log\big(2x\big) - \log\big(2(1-x)\big)$ over $x \in [0,1]$ is zero, which is precisely the value the form of $\tilde{\xi}_{\min}^N$ above yields. This leads us to

$$Q_S\Big(\frac{1}{2}\big(|0\rangle\langle 0| \otimes \varrho_{+T}^{N-1} + |1\rangle\langle 1| \otimes \varrho_{-T}^{N-1}\big)\Big) =$$
$$\frac{1}{2}\Big[S\big(\varrho_{+T}^{N-1}||\xi_{\min+}^{N-1}\big) + S\big(\varrho_{-T}^{N-1}||\xi_{\min-}^{N-1}\big)\Big]$$
$$\doteq \frac{1}{2}\Big(Q_S\big(\varrho_{+T}^{N-1}\big) + Q_S\big(\varrho_{-T}^{N-1}\big)\Big), \text{ (A5)}$$

but, since states $\varrho_{+T}^{N-1} \doteq \Lambda^{PD}(|\Phi_{+T}^{N-1}\rangle)$ and $\varrho_{-T}^{N-1} \doteq \Lambda^{PD}(|\Phi_{-T}^{N-1}\rangle)$ are local-unitarily equivalent, they posses exactly the same amount and type of quantum correlations. Therefore, the last line of (A5) equals $Q_S\big(\Lambda^{PD}(|\Phi_{+T}^{N-1}\rangle)\big)$ and, together with (A2), renders $Q_S\big(\Lambda^{PD}(|\Phi_{+T}^N\rangle)\big) \geq Q_S\big(\Lambda^{PD}(|\Phi_{+T}^{N-1}\rangle)\big)$. Again as in the derivation of (4), iterating this reasoning $N-2$ times one arrives at (A1). □

Appendix B: Robustness law for generic (possibly mixed) graph states

Here, we extend bounds (4) and (A1) first to arbitrary pure graph states, and then to globally-depolarized arbitrary graph states. To encompass both bounds with the same notation, we use in what follows C to denote a generic measure of quantum correlations, wich can either

be an entanglement monotone, $C = E$, or the relative entropy of quantumness, $C = Q_S$, defined in App. A.

For every qubit of any connected N-qubit graph state $|\mathcal{G}^N\rangle$, a measurement in either the Z or the X bases leaves the remaining qubits in a connected $(N-1)$-qubit graph state $|\mathcal{G}^{N-1}\rangle$ (or in a state local-unitarily equivalent to it, depending on the measurrment outcome) [13]. One can thus apply the the same machinery used in the derivations of (4) and (A1) and arrive at the size-independent bound

$$C\big(\Lambda^{PD}(|\mathcal{G}_T^N\rangle)\big) \geq \ldots \geq C\big(\Lambda^{PD}(|\mathcal{G}_T^2\rangle)\big), \quad (B1)$$

for all $N \geq 2$. Here, $|\mathcal{G}_T^2\rangle$ is a two-qubit graph state (local-unitarily equivalent to $|\Phi_{+T}^2\rangle$), and $|\mathcal{G}_T^N\rangle$ is obtained by applying single-qubit Hadamard rotations to some of the qubits in $|\mathcal{G}^N\rangle$ (those corresponding to the above-mentioned X measurements). Finally, the same arguments hold even for imperfect initial states of the form $v|\mathcal{G}^N\rangle\langle\mathcal{G}^N| + (1-v)\frac{\mathbb{1}}{2^N}$, where $0 \leq v \leq 1$ is some visibility. The resulting robustness law is then

$$C\Big(\Lambda^{PD}\big(v|\mathcal{G}_T^N\rangle\langle\mathcal{G}_T^N| + (1-v)\frac{\mathbb{1}}{2^N}\big)\Big) \geq \ldots$$
$$\geq C\Big(\Lambda^{PD}\big(v|\mathcal{G}_T^2\rangle\langle\mathcal{G}_T^2| + (1-v)\frac{\mathbb{1}}{4}\big)\Big). \quad (B2)$$

Appendix C: Asymptotic value of negativity under exact dephasing

Here, we first show that negativity (5) reduces to (6) when $\alpha_Z = 1$, and then that the latter tends to the N-independent value $1 - p$ in the limit $N \to \infty$, for all $0 \leq p < 1$.

First, taking $\alpha_Z = 1$ and $\alpha_X = 0 = \alpha_Y$ in (5) leads, through a simple and straightforward calculation, to

$$\mathcal{N}_{(\alpha_Z=1)} = (1-p)\sum_{\mu=0}^{\lfloor\frac{N'}{2}\rfloor}\binom{N'}{\mu}\Big|\frac{p^\mu}{2}(1-\frac{p}{2})^{N'-\mu}$$
$$-\frac{p}{2}^{N'-\mu}(1-\frac{p}{2})^\mu\Big|, \quad (C1)$$

with $N' \doteq N - 1$. Let us see that the absolute value inside this summation can be removed. Notice that, for all $0 \leq p \leq 1$, and for any $0 \leq \mu \leq \lfloor\frac{N'}{2}\rfloor$, it is

$$1 - \frac{p}{2} \geq \frac{p}{2} \Rightarrow$$
$$(1-\frac{p}{2})^{N'-2\mu} \geq \frac{p}{2}^{N'-2\mu} \Rightarrow$$
$$\frac{p^\mu}{2}(1-\frac{p}{2})^{N'-\mu} \geq \frac{p}{2}^{N'-\mu}(1-\frac{p}{2})^\mu.$$

Therefore $\Big|\frac{p^\mu}{2}(1-\frac{p}{2})^{N'-\mu} - \frac{p}{2}^{N'-\mu}(1-\frac{p}{2})^\mu\Big| \equiv \frac{p^\mu}{2}(1-$

$\frac{p}{2})^{N'-\mu} - \frac{p}{2}^{N'-\mu}(1-\frac{p}{2})^\mu$, what implies

$$\mathcal{N}_{(\alpha_Z=1)} = (1-p)\sum_{\mu=0}^{\lfloor\frac{N'}{2}\rfloor}\binom{N'}{\mu}\Big[\frac{p^\mu}{2}(1-\frac{p}{2})^{N'-\mu}$$
$$-\frac{p}{2}^{N'-\mu}(1-\frac{p}{2})^\mu\Big]. \quad (C2)$$

Next we show that $\lim_{N\to\infty}\mathcal{N}_{(\alpha_Z=1)} = 1 - p$, for all $0 \leq p < 1$. To this end, see first that

$$\sum_{\mu=0}^{\lfloor\frac{N'}{2}\rfloor}\binom{N'}{\mu}\frac{p}{2}^{N'-\mu}(1-\frac{p}{2})^\mu \equiv$$
$$\sum_{\mu=\tilde{N}}^{N'}\binom{N'}{\mu}\frac{p^\mu}{2}(1-\frac{p}{2})^{N'-\mu}, \quad (C3)$$

with

$$\tilde{N} \doteq \begin{cases} \lfloor\frac{N'}{2}\rfloor + 1 & \text{if } N \text{ is even,} \\ \lfloor\frac{N'}{2}\rfloor & \text{if } N \text{ is odd.} \end{cases}$$

Using (C3), one rewrites negativity (6) as $\mathcal{N}_{(\alpha_Z=1)} = (1-p)(S_1 - S_2)$, where S_1 and S_2 are the following sums:

$$S_1 \doteq \sum_{\mu=0}^{\lfloor\frac{N'}{2}\rfloor}\binom{N'}{\mu}\frac{p^\mu}{2}(1-\frac{p}{2})^{N'-\mu}, \quad (C4a)$$

$$S_2 \doteq \sum_{\mu=\tilde{N}}^{N'}\binom{N'}{\mu}\frac{p^\mu}{2}(1-\frac{p}{2})^{N'-\mu}. \quad (C4b)$$

Now, invoking the binomial theorem, we notice that

$$S_1 + S_2 \equiv \begin{cases} 1 & \text{if } N \text{ is even,} \\ 1 + \binom{N'}{N'/2}\frac{p}{2}^{\frac{N'}{2}}(1-\frac{p}{2})^{\frac{N'}{2}} & \text{if } N \text{ is odd.} \end{cases}$$

Therefore, negativity ((6)) can be expressed as $\mathcal{N}_{(\alpha_Z=1)} = (1-p)(1-2S_2)$, when N is even, and as $\mathcal{N}_{(\alpha_Z=1)} = (1-p)\Big(1-2S_2 + \binom{N'}{N'/2}\frac{p}{2}^{\frac{N'}{2}}(1-\frac{p}{2})^{\frac{N'}{2}}\Big)$, when N is odd.

Finally, we show first that $\binom{N'}{N'/2}\frac{p}{2}^{\frac{N'}{2}}(1-\frac{p}{2})^{\frac{N'}{2}} \to 0$ as $N' \to \infty$, and then that also $S_2 \to 0$ as $N' \to \infty$, which finishes the proof. For sufficiently large N', we can apply Stirling's approximation for the factorial: $N'! \to \sqrt{2\pi N'}\frac{N'}{e}^{N'}$. So, we obtain that, for $N' \to \infty$,

$$\binom{N'}{N'/2}\frac{p}{2}^{\frac{N'}{2}}(1-\frac{p}{2})^{\frac{N'}{2}} \to$$
$$\sqrt{\frac{2}{\pi N'}}2^{N'}\frac{p}{2}^{\frac{N'}{2}}(1-\frac{p}{2})^{\frac{N'}{2}} \doteq$$
$$\sqrt{\frac{2}{\pi}}\frac{\gamma^{N'}}{N'} \to$$
$$0,$$

where the convergence of the last limit is guaranteed by the fact that $0 \leq \gamma \doteq 2^2 \frac{p}{2}(1 - \frac{p}{2}) < 1$, for all $0 \leq p \leq 1$.

Now, from definition (C4b) and the facts that $\binom{N'}{\mu} \leq \binom{N'}{N'/2}$ and $\frac{p}{2}^\mu (1 - \frac{p}{2})^{N'-\mu} \leq \frac{p}{2}^{\frac{N'}{2}} (1 - \frac{p}{2})^{\frac{N'}{2}}$, for all $\tilde{N} \leq \mu \leq N'$, we see that sum S_2 is bounded from above as

$$S_2 \leq \begin{cases} \frac{N}{2} \binom{N'}{N'/2} \frac{p}{2}^{\frac{N'}{2}} (1 - \frac{p}{2})^{\frac{N'}{2}} & \text{if } N \text{ is even,} \\ \left(\frac{N}{2} + 1\right) \binom{N'}{N'/2} \frac{p}{2}^{\frac{N'}{2}} (1 - \frac{p}{2})^{\frac{N'}{2}} & \text{if } N \text{ is odd.} \end{cases}$$

Invoking once more Stirling's approximation, using $N' = N - 1$ and the definition of γ above, we have that for large N these bounds are approximately given by

$$S_2 \leq \begin{cases} \sqrt{\frac{1}{2\pi} \frac{N^2}{N-1} \gamma^{N-1}} & \text{if } N \text{ is even,} \\ \sqrt{\frac{1}{2\pi} \frac{(N+2)^2}{N-1} \gamma^{N-1}} & \text{if } N \text{ is odd.} \end{cases}$$

As $N \to \infty$, both bounds tend to the quantity $\sqrt{\frac{1}{2\pi} N \gamma^{N-1}}$, whose limiting value is 0 again because $0 \leq \gamma < 1$. $\square$

[22] A. Streltsov, H. Kampermann, and D. Bruß, Phys. Rev. Lett. **107**, 170502 (2011).